

# Symplectic Quantization of Massive Bosonic string in background B-field

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## Abstract

We give the details of symplectic quantization for a system containing second class constraints. This method is appropriate for imposing infinite series of constraints due to the boundary conditions. We use this method for massive bosonic strings in a background B-field and find the correct expansions of the fields in terms of the physical modes. We have found a canonical basis for this model.

**Keywords:** Symplectic Quantization, Constraints, Bosonic strings

## 1 Introduction

Quantization process has been one of the most important problems from the early days of quantum theory. According to the famous prescription of Dirac[1], given a classical theory with a well behaved bracket in the phase space, the corresponding quantum theory comes out as the result of changing the dynamical variables to operators. Hence, the consistent algebraic structure of classical brackets is an important part of the quantization process. If the system possesses second class constraints one needs to consider Dirac brackets instead of Poisson brackets. However, this procedure should be consistent with the dynamics of the system both in the classical and quantum levels. This demand makes us to investigate the consistency of constraints in the course of the time and finally find a consistent algebra of weakly vanishing brackets among the constraints and the Hamiltonian[2, 3].

On the other hand, we have difficulty with the Dirac method since by restricting the system to live on the constraint surface one may miss the beautiful Poisson structure of the phase space. In other words, we need a new canonical structure that guarantees a closed algebra of Poisson brackets on the constraint surface. According to the famous "Darboux theorem" one can, in principal, find a set of canonical coordinates in which the constraint surface is described by a number of canonical conjugate pairs in addition to some extra coordinates. Faddeev and Jackiw [4] used the above theorem, at each step of consistency, to construct a new method for analyzing the constrained systems. The Faddeev-Jackiw formulation and its equivalence with the Dirac approach is studied in detail by several authors, see for instance [5, 6].

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A modified version of Faddeev-Jackiw formulation recognized as "symplectic quantization" is more or less used by some authors, when they want to build a quantum theory out of a given classical action[7, 8, 9, 10]. As we will see, the most essential point in this process is finding the complete set of physical modes and the correct expansions of the fields in terms of them. The main goal of this paper is giving the essential aspects of symplectic quantization and applying it for quantization of massive bosonic string in a background B-field as a rich example. We will show that the method in its original form works well for the bosonic string in background B-field with no need to any modification such as the additional time averaging suggested in [9].

In this paper we first review the essential aspects of the symplectic quantization in section 2. We restrict ourselves to a purely second class system. This is specially the case when all the constraints have originated from the boundary conditions. In section 3 we consider in details the massive bosonic string again in a background B-field. As we will see the constraint structure of this case is more than a simple extension of the massless case. Then we exhibit the power of the symplectic quantization method in this interesting and illuminating example. The results on the commutation relations would be discussed in the text and also in the last section which is devoted to the concluding remarks.

## 2 The Symplectic Approach

The Hamiltonian formulation of a dynamical system can be achieved by using the first order Lagrangian

$$L_{F.O.} = \sum_i p_i \dot{q}_i - H(q, p). \quad (1)$$

If one uses general coordinates  $\xi^i$  ( $i = 1, \dots, 2N$ ) for the phase space, which may be or may not be canonical, the first order Lagrangian would be written as

$$L_{F.O.} = \sum_i a_i(\xi) \dot{\xi}^i - H(\xi). \quad (2)$$

In this form the action is

$$S = \int [a_i(\xi) d\xi^i - H(\xi) dt], \quad (3)$$

where the kinetic term exhibit a one-form on the phase space manifold. The Euler-Lagrange equations of motion can be written as  $\omega_{ij} \dot{\xi}^j = \partial H / \partial \xi^i$ , where  $\omega_{ij} = \partial a_i / \partial \xi^j - \partial a_j / \partial \xi^i$  is the symplectic matrix. It defines the "symplectic two-form" as

$$\Omega = \frac{1}{2} \omega_{ij} d\xi^i \wedge d\xi^j = da, \quad (4)$$

where  $a$  is the kinetic one-form of the action (3). The equations of motion can be solved as  $\dot{\xi}^i = \omega^{ij} \partial H / \partial \xi^j$ , where  $\omega^{ij}$  is the inverse of  $\omega_{ij}$ . These equations should be equivalent to the canonical equations of motion  $\dot{\xi}^i = \{ \xi^i, H \}$ , where  $\{ , \}$  means the Poisson bracket. Hence, for any two functions  $f(\xi)$  and  $g(\xi)$  the Poisson bracket should be defined as

$$\{f, g\} = \sum_{i,j=1}^N \frac{\partial f}{\partial \xi^i} \omega^{ij} \frac{\partial g}{\partial \xi^j}. \quad (5)$$

As we see, the important role of the symplectic two-form is that *the inverse of symplectic matrix defines the Poisson brackets in arbitrary coordinates of phase space*.

Now, suppose we are given a set of second class constraints  $\phi_a(\xi) \approx 0$  on the phase space variables. Suppose this set includes primary constraints as well as secondary constraints which come out from the consistency of primary ones. In other words, no new constraint may be added to the system as the result of consistency of the present constraints. For instance, these constraints can be considered as the infinite set of boundary conditions and their consistency conditions, to be discussed in the following section. Under imposing the constraints on arbitrary coordinates  $\xi^k$  of the original phase space, the reduced phase space is described by the coordinates  $\eta^a$   $a = 1, \dots, 2m$  ( $m < N$ ), which are not necessarily canonical. On the reduced phase space, we have  $\xi^k = \xi^k(\eta^a)$ , and the induced symplectic tensor is

$$\omega_{ab} = \frac{\partial \xi^i}{\partial \eta^a} \frac{\partial \xi^j}{\partial \eta^b} \omega_{ij}. \quad (6)$$

The induced symplectic tensor  $\omega_{ab}$  can be written by imposing the constraints on the symplectic two-form  $\Omega$ . Inverting  $\omega_{ab}$ , then gives the inverse tensor  $\omega^{ab}$  which determines the Dirac brackets on the reduced phase space.

For a field theory in  $d + 1$  dimensions with the dynamical fields  $\phi_s(x, t)$  the symplectic two-form in terms of the original fields is given by

$$\Omega = \sum_s \int d^d x d\Pi_s(x, t) \wedge d\phi_s(x, t). \quad (7)$$

where  $\Pi(x, t)$  are momentum fields. Upon imposing a set of second class constraints, such as initial boundary conditions and their consistency conditions, suppose the phase space fields  $\phi_s(x, t)$  and  $\Pi_s(x, t)$  can be written in terms of a restricted set of physical mods  $a_n(t)$ . Inserting the corresponding expansions of  $\phi_s(x, t)$  and  $\Pi_s(x, t)$  in Eq. (7), the symplectic two-form can be written, in principal, as

$$\Omega = \sum_{n,m} \omega_{nm} da^n \wedge da^m. \quad (8)$$

The summations in (8) are understood to include integration over the continues variables in cases where physical modes  $a_n(t)$  depend on continues parameters. The inverse tensor  $\omega^{nm}$  defines the Dirac brackets of the reduced phase space coordinates as

$$\{a^n, a^m\} = \omega^{nm}. \quad (9)$$

Note that the initial Poisson brackets is no longer valid on the reduced phase space.

We emphasize the important point that up to this point the symplectic quantization is done without solving the equations of motion of the variables or fields. In other words, we have used the first part of the action (3) so far. The dynamics is, however, the responsibility of the second part, i.e. the Hamiltonian. Properties of the Poisson structure on the classical phase space, as well as the algebra of the operators in quantum mechanics do not depend at all on the form of the Hamiltonian. For example for a particle in one dimension the most important thing for quantization is the classical bracket  $\{x, p\} = 1$  or equivalently  $\{a, a^*\} = \frac{1}{i\hbar}$ ; which converts to  $[x, p] = i\hbar$  or  $[a, a^\dagger] = 1$  upon quantization. These relations do not depend on the explicit time dependence of  $a(t)$  and  $a^\dagger(t)$  to be for example  $a(0)e^{i\omega t}$  and  $a^\dagger(0)e^{-i\omega t}$  for simple harmonic oscillator.

It worths reminding that upon ordinary conditions, the classical brackets, as well as the quantum commutators, are independent of time. To see this explicitly, suppose at time  $t$  we have  $\{\xi^i(t), \xi^j(t)\} = \omega^{ij}$ . Assume  $\omega^{ij}$  is independent of  $\xi^i, s$ , which is the case in most of the problems. In the course of the time, the variation of  $\xi^i$  is  $\delta\xi^i = \{\xi^i, H\}\delta t$ . Using the Jacobi identity the bracket of  $\xi^i$  and  $\xi^j$  at time  $t + \delta t$  is

$$\begin{aligned} \{\xi^i + \delta\xi^i, \xi^j + \delta\xi^j\} &= \omega^{ij} + \delta t [\{\{\xi^i, H\}, \xi^j\} + \{\xi^j, \{\xi^i, H\}\}] \\ &= \omega^{ij} + \delta t \{H, \{\xi^i, \xi^j\}\} \\ &= \omega^{ij}. \end{aligned} \tag{10}$$

### 3 Massive bosonic string

Consider the Lagrangian of the massive bosonic string in an external B-field,

$$L = \frac{1}{2} \int_0^l d\sigma \left[ \dot{X}^2 - X'^2 - m^2 X^2 + 2B_{ij} \dot{X}_i X'_j \right], \tag{11}$$

where "dot" and "prime" represent differentiation with respect to  $\tau$  and  $\sigma$  respectively and  $X^2 \equiv X_i X_i$ , etc. This is the simplified version of a model in which among the whole set of bosonic fields  $X^\mu$  an even number  $X^i$  are influenced by a background antisymmetric B-field and are attached at the end-points to D-brains. Here, we have omitted those components of  $X^\mu$  which possess ordinary Neumann boundary conditions and are not coupled to B-field. We consider the simplest case where  $i = 1, 2$ , and the metric of the truncated target space is Euclidian. Hence, we consider all the space indices as lower indices. The antisymmetric B-field in two space dimensions can be written as

$$B_{ij} \equiv \begin{pmatrix} 0 & \tilde{B} \\ -\tilde{B} & 0 \end{pmatrix}. \tag{12}$$

The massless case ( $m = 0$ ) is studied in different aspects by several authors [11, 12, 13, 14, 15] with the well-known result of non commutativity at the end-points. Demanding the variation of the action vanish under arbitrary variation  $\delta X_i$  gives the equation of motion as  $(\partial_\tau^2 - \partial_\sigma^2 - m^2) X_i = 0$ , while the boundary conditions are  $X'_i + B_{ij} \dot{X}_j = 0$  at  $\sigma = 0$  and  $\sigma = l$ . The momentum fields are given by  $P_i = \dot{X}_i + B_{ij} X'_j$ . The canonical Hamiltonian reads

$$H = \frac{1}{2} \int_0^l d\sigma [(P - BX')^2 + X'^2 + m^2 X^2]. \tag{13}$$

The fundamental Poisson brackets (before imposing the constraints) read

$$\begin{aligned} \{X_i(\sigma, \tau), X_j(\sigma', \tau)\} &= 0, \\ \{P_i(\sigma, \tau), P_j(\sigma, \tau)\} &= 0, \\ \{X_i(\sigma, \tau), P_j(\sigma, \tau)\} &= \delta_{ij} \delta(\sigma - \sigma'). \end{aligned} \tag{14}$$

Consider the boundary conditions as primary constraint  $\Phi_i^{(1)} \equiv \phi_i^{(1)}|_{\sigma=0}$  and  $\bar{\Phi}_i^{(1)} \equiv \phi_i^{(1)}|_{\sigma=l}$  where

$$\phi_i^{(1)} \equiv M_{ij} X'_j + B_{ij} P_j. \tag{15}$$

in which  $M = 1 - B^2$ . As indicated in details in references [11, 12], consistency of primary constraints in time determines the Lagrange multipliers  $\lambda_i$  and  $\bar{\lambda}_i$  in the total Hamiltonian  $H_T = H_c + \lambda_i \Phi_i^{(1)} + \bar{\lambda}_i \bar{\Phi}_i^{(1)}$  to be zero and at the same time gives second level constraints as  $\Phi_i^{(2)} \equiv \psi_i(0, \tau)$  and  $\bar{\Phi}_i^{(2)} \equiv \psi_i(l, \tau)$  where

$$\psi_i(\sigma, \tau) = \partial_\sigma P_i - m^2 B_{ij} X_j. \quad (16)$$

Direct calculation shows that third and forth level constraints are  $(\partial_\sigma^2 - m^2)\phi_i$  and  $(\partial_\sigma^2 - m^2)\psi_i$  at the end-points, respectively. In this way even and odd level constraints at  $\sigma = 0$  read respectively as

$$\begin{aligned} \Phi_i^{(2r)} &= (\partial_\sigma^2 - m^2)^{r-1} \psi_i |_{\sigma=0} \approx 0 \quad r \geq 1, \\ \Phi_i^{(2r+1)} &= (\partial_\sigma^2 - m^2)^r \phi_i |_{\sigma=0} \approx 0 \quad r \geq 0, \end{aligned} \quad (17)$$

with similar expressions for  $\bar{\Phi}_i^{(2r)}$  and  $\bar{\Phi}_i^{(2r+1)}$  at  $\sigma = l$ .

Expanding the real fields  $X(\sigma, \tau)$  and  $P(\sigma, \tau)$  as Fourier integrals, we have

$$\begin{aligned} X_i(\sigma, \tau) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk [a_i(k, \tau) \cos k\sigma + b_i(k, \tau) \sin k\sigma], \\ P_i(\sigma, \tau) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk [c_i(k, \tau) \cos k\sigma + d_i(k, \tau) \sin k\sigma]. \end{aligned} \quad (18)$$

It is clear that  $a_i(k, \tau)$  and  $c_i(k, \tau)$  are even while  $b_i(k, \tau)$  and  $d_i(k, \tau)$  are odd functions of  $k$ . Before imposing the constraints all of the above modes are present. Using the fundamental brackets (14) one can find the following Poisson brackets among the Fourier modes

$$\begin{aligned} \{a_i(k, \tau), c_j(k', \tau)\} &= \delta_{ij} \delta(k - k'), \\ \{b_i(k, \tau), d_j(k', \tau)\} &= \delta_{ij} \delta(k - k'), \end{aligned} \quad (19)$$

while all other bracket vanish. This shows that the canonical pairs  $(a_i(k, \tau), c_i(k, \tau))$  and  $(b_i(k, \tau), d_i(k, \tau))$  act as an alternative canonical basis for the original phase space. Upon imposing the constraints a large number of these modes will be omitted, leaving a much smaller number of them as the canonical coordinates of the reduced phase space.

We now impose the set of constraints (17) on the Fourier expansions (18) to find

$$\begin{aligned} \int_{-\infty}^{\infty} dk (-1)^r (k^2 + m^2)^r [B_{ij} c_j(k, \tau) + k M_{ij} b_j(k, \tau)] &= 0, \\ \int_{-\infty}^{\infty} dk (-1)^r (k^2 + m^2)^r [k d_i(k, \tau) - m^2 B_{ij} a_j(k, \tau)] &= 0. \end{aligned} \quad (20)$$

Since the expressions in the brackets are even functions of  $k$ , the conditions (20) for arbitrary  $r$  will be satisfied only if  $b(k, \tau) = -M^{-1} B c(k, \tau)/k$  and  $d(k, \tau) = m^2 B a(k, \tau)/k$ . This leads to omitting the modes  $b(k, \tau)$  and  $d(k, \tau)$  in favor of  $a(k, \tau)$  and  $c(k, \tau)$  respectively. For simplicity we omit the indices  $i, j, \dots$  on the fields and modes from now on. Imposing the constraints at the end point  $\sigma = l$  gives

$$\begin{aligned} \int_{-\infty}^{\infty} dk (-1)^r (k^2 + m^2)^r \left[ \frac{1}{k} m^2 B^2 - k M \right] a(k, \tau) \sin kl &= 0, \\ \int_{-\infty}^{\infty} dk (-1)^r (k^2 + m^2)^r \left[ -k + \frac{1}{k} m^2 B^2 M^{-1} \right] c(k, \tau) \sin kl &= 0. \end{aligned} \quad (21)$$

These equations show that  $a(k, \tau)$  and  $c(k, \tau)$  should vanish except for  $k = n\pi/l$  with integer  $n$  or when

$$k^2 = m^2 B^2 M^{-1} = -\frac{m^2 \tilde{B}^2}{1 + \tilde{B}^2} \equiv -k_0^2. \quad (22)$$

The first possibility leads to oscillatory modes as

$$\begin{aligned} X_{\text{os}} &= \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left[ a^{(n)}(\tau) \cos \frac{n\pi\sigma}{l} - \frac{l}{n\pi} M^{-1} B c^{(n)}(\tau) \sin \frac{n\pi\sigma}{l} \right], \\ P_{\text{os}} &= \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left[ c^{(n)}(\tau) \cos \frac{n\pi\sigma}{l} + \frac{l}{n\pi} m^2 B a^{(n)}(\tau) \sin \frac{n\pi\sigma}{l} \right]. \end{aligned} \quad (23)$$

The normalization coefficients  $\sqrt{2/l}$  are chosen for future convenience. This choice also makes the correct dimensionality for  $a^{(n)}$  as  $(\text{Mass})^{-1/2}$  and for  $c^{(n)}$  as  $(\text{Mass})^{1/2}$ . Note that the fields  $X_i$  are dimensionless.

The possibility (22) corresponds to zero mode solutions with  $\sinh k_0\sigma$  and  $\cosh k_0\sigma$ . Traditionally the zero mode solution is denoted as the zero frequency (infinite wave length) limiting term in the Fourier expansions, as shown for example for the massless case of the current problem in [11]. Here, however, we interpret *the zero mode solution as a solution which satisfies the boundary conditions not only at the end-points but also throughout all the medium*. This interpretation works well for constant term in the case of ordinary Neumann boundary condition of free bosonic string as well as for the zero mode terms for the massless string in B-field. To see the details, suppose the first and second level constraints are valid throughout all the string as the following coupled differential equations

$$\begin{aligned} BP + MX' &= 0, \\ P' - m^2 BX &= 0, \end{aligned} \quad (24)$$

which gives  $MX'' + m^2 B^2 X = 0$ . Since  $B^2 = -\tilde{B}^2 \mathbf{1}$  and  $M = (1 + \tilde{B}^2)\mathbf{1}$  the most general solutions of Eqs. (24) can be chosen as

$$\begin{aligned} X_{\text{zm}}(\sigma, \tau) &= \frac{1}{\sqrt{l}} \left[ a^{(0)}(\tau) \cosh[k_0(\sigma - l/2)] - \frac{1}{k_0} M^{-1} B c^{(0)}(\tau) \sinh[k_0(\sigma - l/2)] \right] \\ P_{\text{zm}}(\sigma, \tau) &= \frac{1}{\sqrt{l}} \left[ c^{(0)}(\tau) \cosh[k_0(\sigma - l/2)] + \frac{1}{k_0} m^2 B a^{(0)}(\tau) \sinh[k_0(\sigma - l/2)] \right] \end{aligned} \quad (25)$$

where  $k_0$  is given in Eq.(22). Again we have imposed the normalization coefficient  $\sqrt{1/l}$  for making correct dimensionality and future convenience. Note that we have chosen the argument of sinh and cosh functions as measured from the center of the string. Besides respecting the symmetry of the Hamiltonian under  $\sigma \rightarrow -\sigma$  when we shift the origin of  $\sigma$  to  $l/2$ , the reason for this choice is that the physical modes in this sector turn out finally to be canonical modes, as we will see in the near future. The most general solution of the fields are superposition of zero-mode and oscillatory solutions as

$$\begin{aligned} X &= X_{\text{zm}} + X_{\text{os}}, \\ P &= P_{\text{zm}} + P_{\text{os}}. \end{aligned} \quad (26)$$

Now we want to quantize the theory using the symplectic approach. After a long but direct calculation the symplectic two-form  $\Omega = \sum_i \int d\sigma dP_i(\sigma, \tau) \wedge dX_i(\sigma, \tau)$  in terms of the physical modes  $a^{(n)}(\tau)$  and  $c^{(n)}(\tau)$  read

$$\Omega = \frac{\sinh k_0 l}{l k_0} \left( dc_i^{(0)} \wedge da_i^{(0)} \right) + \sum_{n=1}^{\infty} \left( 1 + \frac{l^2 k_0^2}{n^2 \pi^2} \right) \left( dc_i^{(n)} \wedge da_i^{(n)} \right) \quad (27)$$

The important point in derivation of Eq.(27) is that all cross terms composed of zero-modes wedge oscillatory modes have been canceled. This is a good news, since it makes the symplectic matrix block diagonal with one  $4 \times 4$  block for the zero modes and an infinite dimensional block for the oscillatory modes. Moreover in oscillatory modes there is no mixing between different oscillators (i.e. different  $n$ ). In zero mode sector also we do not encounter cross terms such as  $da_i^{(0)} \wedge da_j^{(0)}$  and  $dc_i^{(0)} \wedge dc_j^{(0)}$ . This is because of our suitable choice of zero modes in Eqs (25) as combinations of  $\sinh[k_0(\sigma - l/2)]$  and  $\cosh[k_0(\sigma - l/2)]$ . These opportunities make it easy to find the inverse of symplectic matrix and write down the brackets of physical modes. Hence, the non-vanishing Dirac brackets among physical modes are as follows

$$[a_i^{(n)}, c_j^{(s)}] = N_n^{-1} \delta_{ij} \delta_{ns}, \quad (28)$$

where

$$N_0 \equiv \frac{\sinh(k_0 l)}{k_0 l}, \quad N_n \equiv 1 + \frac{k_0^2 l^2}{n^2 \pi^2} \quad n \neq 0. \quad (29)$$

Note that we did not need to solve the equations of motion up to this point. The theory can be quantized, without any need to full solutions of the equations of motion by converting the brackets (28) into the quantum commutators. Similar to the massless case, one can find the time dependence of the physical modes by directly solving their equations of motion. For this aim it is more economical to write the canonical Hamiltonian (13) in terms of the physical modes. The result is

$$H = \frac{1}{2} \sum_i \sum_{n=0}^{\infty} N_n \left[ M^{-1} [c_i^{(n)}]^2 + M \omega_n^2 [a_i^{(n)}]^2 \right], \quad (30)$$

where

$$\omega_0^2 \equiv m^2 M, \quad \omega_n^2 \equiv m^2 + \frac{n^2 \pi^2}{l^2} \quad n \neq 0 \quad (31)$$

The Hamiltonian (30) is a superposition of infinite number of independent harmonic oscillators. The canonical equations of motion can be solved for  $n = 0, 1, \dots$  as

$$\begin{aligned} a^{(n)}(\tau) &= a^{(n)}(0) \cos \omega_n \tau + \frac{1}{M \omega_n} c^{(n)}(0) \sin \omega_n \tau, \\ c^{(n)}(\tau) &= c^{(n)}(0) \cos \omega_n \tau - M \omega_n a^{(n)}(0) \sin \omega_n \tau, \end{aligned} \quad (32)$$

where  $a^{(n)}(0)$  and  $c^{(n)}(0)$  are Schrodinger modes. Inserting  $a^{(n)}(\tau)$  and  $c^{(n)}(\tau)$  from Eqs. (32) in the expansions (23) and (25) of the fields, gives in fact the solutions of the equations of motion consistent with the boundary conditions, written in terms of the canonical Schrodinger modes.

Our results here are different from those of reference [10] in the following aspects:

- 1) The algebra of physical modes  $a^{(n)}(\tau)$  and  $c^{(n)}(\tau)$  is much simpler. They are canonical conjugate pairs with position-momentum like brackets given in (28).

2) Inserting our final expansions of the fields in terms of the Schrodinger modes in the symplectic two-form leads again to expression (27) for the initial values of the modes. In other words, there is no explicit time dependence as expected. However, direct calculation shows that, similar to the massless case of reference [9], the symplectic two-form constructed from expansions given in [10] contains time dependent terms.

It seems that real functions for spatial dependence of the fields should be accompanied naturally with real time dependent of the modes as seen in (32). Combining spatial real functions with the time dependence of the form  $e^{i\omega\tau}$  may lead to unwanted time dependence in the symplectic two-form. In fact, this time dependence is the origin of an unnecessary step of time averaging of the symplectic two-form suggested in reference [9].

In order to complete our discussions and enlighten some unclear points of the literature let us calculate the Dirac brackets of the original fields to see the effect of the B-field and boundary conditions on the commutativity of fields. To do this using the brackets (28) and Eqs.(23) ,(25) and (26) we can compute the equal time Dirac brackets of the original coordinates and momentum fields as follows

$$[X_i(\sigma, \tau), X_j(\sigma', \tau)] = 2(BM^{-1})_{ij}f(\sigma + \sigma') , \quad (33)$$

$$[P_i(\sigma, \tau), P_j(\sigma', \tau)] = 2m^2B_{ij}f(\sigma + \sigma') , \quad (34)$$

$$[X_i(\sigma, \tau), P_j(\sigma', \tau)] = \delta_{ij}g(\sigma, \sigma') \quad (35)$$

where

$$f(\sigma) = \frac{\sinh[k_0(\sigma - l)]}{\sinh k_0 l} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 + \frac{l^2 k_0^2}{n^2 \pi^2}\right)^{-1} \sin\left(\frac{n\pi\sigma}{l}\right) , \quad (36)$$

and

$$\begin{aligned} g(\sigma, \sigma') &= 2k_0 \frac{\cosh[k_0(\sigma + \sigma' - l)]}{\sinh k_0 l} \\ &\quad + \frac{4}{l} \sum_{n=1}^{\infty} \left(1 + \frac{l^2 k_0^2}{n^2 \pi^2}\right)^{-1} \cos \frac{n\pi\sigma}{l} \cos \frac{n\pi\sigma'}{l} \\ &\quad + \frac{4}{l} \sum_{n=1}^{\infty} \frac{l^2 k_0^2}{n^2 \pi^2} \left(1 + \frac{l^2 k_0^2}{n^2 \pi^2}\right)^{-1} \sin \frac{n\pi\sigma}{l} \sin \frac{n\pi\sigma'}{l}. \end{aligned} \quad (37)$$

Let us first consider the brackets (33) and (34) in details. The function  $f(\sigma)$  reduces, for  $m = 0$ , to

$$f_0(\sigma) = -1 + \frac{\sigma}{l} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi\sigma}{l}\right) , \quad (38)$$

which gives

$$\begin{aligned} \{X_i(\sigma, \tau), X_j(\sigma', \tau)\} &= 0 \quad \sigma, \sigma' \neq 0, \\ \{X_i(0, \tau), X_j(0, \tau)\} &= -2(M^{-1}B)_{ij}, \\ \{X_i(l, \tau), X_j(l, \tau)\} &= 2(M^{-1}B)_{ij}. \end{aligned} \quad (39)$$

which is the standard results of non commutativity of the end-points of a massless string in the background B-field. For  $m \neq 0$  using the Fourier expansion

$$\sinh[k_0(\sigma - l)] = \frac{2 \sinh k_0 l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{n^2 \pi^2}{n^2 \pi^2 + k_0^2 l^2}\right) \sin\left(\frac{n\pi\sigma}{l}\right) , \quad (40)$$

in the interval  $[0, 2l]$ , it is easily seen that  $f(\sigma)$  vanishes for every value of  $\sigma$  in the open interval  $(0, 2l)$ . For  $\sigma = 0$  and  $\sigma = 2l$ , however, the Fourier expansion (40) is no longer valid since, beyond the interval  $(0, 2l)$ , the hyperbolic function on the left hand side of Eq. (40) can not be expanded in terms of periodic functions  $\sin(n\pi\sigma/l)$ . Similarly, the last expression in the right hand side of Eq. (38) is the expansion of the non-continues sawtooth function with non-continuity on the points  $\sigma = 2kl$  for integer  $k$ . At the end-points  $\sigma = 0$  and  $\sigma = 2l$  the function  $f_0(\sigma)$  is +1 and -1 respectively from its definition in Eq. (36). The above argument can be used exactly in the same way for  $f(\sigma)$  given in Eq. (36). In other words,  $f(\sigma)$  in Eq. (36), for arbitrary  $k_0$ , is the same as given in Eq. (38) for the case  $k_0 = 0$ . Hence we have

$$f(\sigma) = 0 \text{ for } 0 < \sigma < 2l, \quad f(0) = -1, \quad f(2l) = 1. \quad (41)$$

This result shows that non-commutativity of the coordinate fields at the end points are exactly the same for massless and massive cases.

As we see from Eqs. (33) and (34), for  $m \neq 0$ , the momentum fields as well as the coordinate fields are noncommutative at the end-points. This is due to dependence of momentum fields to coordinate Fourier modes  $a_i^{(n)}$  in contrast to massless case which is just composed of momentum Fourier modes  $c_i^{(n)}$ . It worth remembering that mixing of coordinate and momentum Fourier modes in the expansions of  $X(\sigma, \tau)$  and  $P(\sigma, \tau)$  is basically due to mixed boundary conditions which is in turn resulted from the presence of B-field.

Next we consider the bracket of a coordinate and a momentum field in Eq. (35). The function  $g(\sigma, \sigma')$  given in Eq. (37) can be written as

$$g(\sigma, \sigma') = \frac{4}{l} \sum_{n=1}^{\infty} \cos \frac{n\pi\sigma}{l} \cos \frac{n\pi\sigma'}{l} + \bar{g}(\sigma + \sigma'), \quad (42)$$

where

$$\bar{g}(\sigma) = 2k_0 \frac{\cosh[k_0(\sigma - l)]}{\sinh k_0 l} - \frac{4}{l} \sum_{n=1}^{\infty} \left( \frac{l^2 k_0^2}{n^2 \pi^2 + l^2 k_0^2} \right) \cos \frac{n\pi\sigma}{l}. \quad (43)$$

Using the Fourier expansion

$$\cosh[k_0(\sigma - l)] = \frac{\sinh k_0 l}{k_0 l} + \sum_{n=1}^{\infty} \left( \frac{2k_0 l \sinh k_0 l}{n^2 \pi^2 + k_0^2 l^2} \right) \cos \left( \frac{n\pi\sigma}{l} \right), \quad (44)$$

for  $\sigma \in [0, 2l]$ , it is easily seen that  $\bar{g}(\sigma) = 2/l$ . So we have finally

$$g(\sigma, \sigma') = \frac{2}{l} + \frac{4}{l} \sum_{n=1}^{\infty} \cos \frac{n\pi\sigma}{l} \cos \frac{n\pi\sigma'}{l} = \delta(\sigma - \sigma'). \quad (45)$$

The interesting point is that the result is again independent of  $m$  or  $k_0$ . In other words, the fields  $X(\sigma, \tau)$  and  $P(\sigma, \tau)$  are still canonical conjugate pairs in the reduced phase space, for the massive case.

## 4 Concluding Remarks

Our main goal in this paper is to reintroduce "the symplectic quantization method", in such a way that is appropriate for imposing the second class constraints originated from the

boundary conditions. We showed that the essential point is imposing the set of constraints on some "appropriate expansion" of the fields in terms of suitable modes. Although the full dynamics of the system is not required in order to use the machinery of the symplectic approach, it is necessary to investigate the dynamics of the constraints; which is, in fact, the program of finding the constraint structure of the system.

The main outcome of this procedure is recognizing all independent physical degrees of freedom, i.e. the coordinates of the reduced phase space, and finding expansions of quantities of interest, such as the original fields and the Hamiltonian in terms of them. Note that in this viewpoint the mode expansions of the fields are not just combinations of the solutions of the equations of motion with some meaningless coefficients. These coefficients, if precisely chosen, should coincide with the initial values of the physical modes (i.e. the Schrodinger modes). Clearly, we would be more happy if these coordinates constitute a canonical basis for the reduced phase space.

We then considered the model of a massive string in a background B-field as an interesting and nontrivial example for applying the idea of considering boundary conditions as Dirac constraints as well as using the symplectic approach for quantizing the model. Besides characteristics such as arising mixed boundary conditions (Eqs. 15 and 16) and the appearance of two sets of infinite number of constraints at each boundary (Eq. 17), which is common with the massless case, the problem of massive bosonic string has its special attractions for two following reasons;

i) The boundary conditions (15) and (16) incorporate coordinate and momentum fields almost on the same footing. This results to expansions (23) and (25), in which canonical conjugate pairs are present both in  $X(\sigma, \tau)$  and  $P(\sigma, \tau)$ , which finally leads to noncommutative momentum fields (see Eq. 34) as opposed to the massless case.

ii) Upon imposing the constraints, the massive bosonic fields acquire nontrivial zero modes (see Eqs. 25) which can not be derived from the limiting case of oscillatory modes. We suggest to define the zero mode as solution of generalization of the boundary conditions to the whole medium, instead of the boundaries alone. Such a solution clearly satisfies the required conditions at the boundaries and should be included in the most possible expansions of the fields. We showed that this inclusion could be happen naturally and need not to be added by hand (see our discussion after Eq. 21).

Our main result for this model is finding a canonical basis for the reduced phase space with a canonical algebra given in (28), which apart from being physically meaningful, is much simpler to work with, compared to that of reference [10]. We showed also that the Hamiltonian of the system is simply the superposition of harmonic oscillators constructed over these coordinate-momentum pairs.

Giving all technical details, we showed that the coordinate and momentum fields are non-commutative at the end-points (Eqs. 33-34 and 41); while they remain, after imposing the constraints, canonically conjugate to each other in the bulk of string (Eqs. 35 and 45).

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